§3 Let $m(E) \leq +\infty$. In §2 we have treated $\int_E f$ $\forall f \in \mathcal{B}_0(E)$, the class of all <u>bounded</u> measurable functions on $E$ vanishing outside some subset of $E$ with finite measure. This section will deal with measurable functions $\geq 0$ (to be denoted by $\mathcal{M}^+(E)$ or $\mathcal{MF}^+(E)$ for the class of all such functions).

Define, $\forall f \in \mathcal{MF}^+(E)$

$$\int_E f \overset{\text{def}}{=} \sup\left\{\int_E h : h \in \mathcal{B}_0(E),\ h \leq f \text{ on } E\right\}$$

$$= \sup\left\{\int_E h : h \in \mathcal{B}_0(E),\ 0 \leq h \leq f \text{ on } E\right\}$$

($0 \leq h \vee 0 \leq f$ on $E$ and $h \vee 0 \in \mathcal{B}_0(E)$ whenever $h \in \mathcal{B}_0(E)$ with $h \leq f$ on $E$).

Note 1 "on $E$" can be replaced by "a.e. on $E$"

Note 2

$$+\infty \geq \int_E f \geq 0$$

Note 3 $\int_E \alpha f = \alpha \int_E f$ $\forall \alpha \geq 0$

MT $f \mapsto \int_E f$ is $\uparrow$ and additive.

The additivity follows from the following

Riesz Lemma. Let $0 \leq l \leq f + g$ on $E$ with $l \in \mathcal{B}_0(E)$ and $f, g \in \mathcal{MF}^+(E)$. Then $\exists$ $h, k \in \mathcal{B}_0(E)$ such that $l = h + k$ and $0 \leq h \leq f$, $0 \leq k \leq g$.

Pf of Lemma. Take $A \subseteq E$ with $m(A) < +\infty$ such that $l = 0$ on $E \setminus A$. Define $h, k : E \to [0, \infty)$ by $h := l \wedge f$ and $k := l - h = l - (l \wedge f) = 0 \vee (l - f)$ (ptwisely on $E$). Then $0 \leq h \leq f$ and $0 \leq k \leq g$ on $E$ and $l = h + k$ on $E$ (in particular $0 \leq h, k \leq l$ on $E$ and so $h, k = 0$ on $E \setminus A$). It is now clear that $h, k \in \mathcal{B}_0(E)$.

Ex1. Let $A \subseteq E$ be measurable and $f \in \mathcal{MF}^+(E)$. Show that the following two natural definitions of $\int_A f$ coincide:

$$\int_E f \chi_A = \sup\left\{ \int_A h : h \in \mathcal{B}_0(A),\ h \leq f \text{ on } A \right\}$$

Ex2. $A \mapsto \int_A f$ is additive and $\uparrow$.

Th 2. (Fatou's Lemma). Let $f_n, f \in \mathcal{MF}^+(E)$ s.t. $f_n \to f$ (ptwise) on $E$. Then $\int_E f \le \liminf_n \int_E f_n$.

Proof. Let $h \in \mathcal{B}_0(E)$ s.t. $0 \le h \le f$. Suff to show that $\int_E h \le \liminf_n \int_E f_n$. Let $h_n := h \wedge f_n$ $\forall n$. Then $0 \le h_n \le f_n$ on $E$ and $h_n \in \mathcal{B}_0(E)$ (any bound of $h$ is also a bound of $h_n$, and if $h = 0$ on $E \setminus H$ with $m(H) < +\infty$ then $h_n = 0$ on $E \setminus H$). Pointwisely on $E$

$$\lim_n h_n = h \wedge (\lim_n f_n) = h \wedge f = h$$

By the <u>generalized</u> bounded MCT, it follows that $\int_E h_n \to \int_E h$. Since $\int_E h_n \le \int_E f_n$ $\forall n$, it follows that $\int_E h \le \liminf_n \int_E f_n$.

Th 3. Monotone Convergence Th. Suppose $0 \le f_n \uparrow f$ ptwise on $E$ and each $f_n \in \mathcal{MF}^+(E)$. Then

$$\lim_n \int_E f_n = \int_E f \qquad (*)$$

pf By Monotonicity of integrals $\int_E f_n \le \int_E f$ $\forall n$. Hence $\limsup_n \int_E f_n \le \int_E f$ $\left(\le \liminf_n \int_E f_n \text{ by Fatou}\right)$, so all equal and $(*)$ holds.

Th 4. $A \mapsto \int_A f$ countably additive

Pf. Since $\chi_{\bigcup_{i=1}^{n} A_i} \uparrow_n \chi_{\bigcup_{i\in\mathbb{N}} A_i}$ for countable disjoint union, and $f \geq 0$ it follows from Th 3 that

$$\int_E f\chi_{\bigcup_{i=1}^{n} A_i} \to \int_E \left(f\chi_{\bigcup_{i\in\mathbb{N}} A_i}\right) = \int_{\bigcup_{i\in\mathbb{N}} A_i} f$$

$$\int_E \left(\sum_{i=1}^{n} f\chi_{A_i}\right) = \sum_{i=1}^{n} \int_{A_i} f$$

Th 5. $A \mapsto \int_A f$ is absolutely cts: $\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t. $\left|\int_A f\right| < \varepsilon$ whenever $A \subseteq E$ with $m(A) < \delta$.

§4. $m(E) \leq +\infty$. $f \in \mathcal{MF}(E)$ is said to be Lebesgue measurable if $\int_E f^+, \int_E f^- < +\infty$

(in this case one says that $f$ is Lebesgue integrable (in notation $f \in \mathcal{L}(E)$ or $\mathcal{L}_1(E)$) and

$$\int_E f := \int_E f^+ - \int_E f^- \quad (\in \mathbb{R}).$$

Notes. $f_1 \sim f_2$ on $E$ $\Rightarrow \int_E f_1 = \int_E f_2$

$f \mapsto \int_E f$ is $\uparrow$ and linear, ~~absolutely cts~~ or "converse of monotonicity"

$A \mapsto \int_A f$ absolutely cts, countably additive.